Analysis of Random and Burst Error Codes in 2-state Markov Channels

Gerhard Haßlinger and Oliver Hohlfeld

***Abstract*—Markov chains are a popular means to capture correlated random processes for characterization and analysis of error pattern or bursty traffic in data transmission. The performance evaluation of communication protocols based on Markov models is tractable with limited state space and often depends on efficient implementation. Simulation or other methods have to be adopted for system of higher complexity.**

We study the effect of bit errors on transmitted packets for Markov error processes in the presence of error detecting or error correcting codes. We show that the evaluation for 2-state Markov channels leads the same 2-state channel characteristics also for packet errors, where parameters are determined by a recursive and closed form solution depending on the parameters of the bit error channel. Computation schemes are derived for random and burst error detecting and correcting codes whose computation effort is shown to be moderate and generally tractable for usual packet sizes in telecommunication.

***Keywords*—2-state Markov processes, error detection, random and burst error correcting codes, residual error rate.**

I. Introduction

MARKOV processes are applied in many studies to investigate the impact of errors in binary data transmission on the performance of coding and retransmission schemes. Fifty years ago, Gilbert and Elliott [2][3] introduced Markov channels distinguishing "Good" phases with no or few sporadic errors occurring at small error probability from "Bad" phases with much higher error probability. The Gilbert-Elliott model is popular for generating and analyzing bit error processes [1][5][11][13][14][15][16] in communication channels. However, the feasibility of a simple analysis is questioned for packets of transmission protocols above the bit stream layer, as expressed e.g. by Khalili and Salamatian [6]:

"The second type of approaches have described the error process as Markov models, for example the well-known Gilbert-Elliott model, but these descriptions failed in providing closed form formula based on code parameters that will enable to design communication systems."

On the contrary, work by Turin et al. [13][14] provides a general analysis for *N*-state "hidden" Markov models in terms of eigenvalue/eigenvector solutions for error burst and gap length distributions as well as coding efficiency. The same method is often applied to discrete Markovian queueing systems [7][10]. However, the reduction of this framework to the case of 2-state models makes the derivation and solution form essentially simpler and better comprehensible. Examples of the 2-state solution in literature are restricted to the Gilbert model [1][13], although the same solution is still valid for the Gilbert-Elliot model and even for a general 2-state channel format with two more parameters as shown in Figure 4.

In this setting, the paper contributes by modeling the performance of error correcting codes under random and burst errors. In particular, closed-form expressions are derived for 2-state Markov channels and the trade off between the performance and overhead of coding is evaluated.

The paper is structured as follows: In Section II we briefly summarize the result for 2-state packet error processes, whose basic implications are presented in [4] in more detail. As the main focus of this work, residual error rates for random and burst error correction and detection are derived in Section III as well as the evaluation for comparison of coding schemes with regard to their overhead in Section IV. Finally, conclusions and extensions for further study are addressed.

II. Packet Errors in 2-State Markov Channels

A. Definitions and Basic Result

We assume bit errors to be generated by a 2-state Markov channel with bitwise state transition, which are characterized by six parameters

q: probability to change from "Good" (G) to "Bad" (B) state,

p: probability to change from the "Bad" to the "Good" state,

h_{GG}: bit error probability in state G, when the current and the next state is "Good",

h_{BB}: bit error probability in state B, when the current and the next state is "Bad",

h_{GB}: bit error probability while changing from state G to B for the next bit,

h_{BG}: bit error probability while changing from state B to G for the next bit.

The popular Gilbert-Elliott model is a special case of those definitions for the general 2-state format where $h_{GG} = h_{GB}$ ($= h_G$) and $h_{BB} = h_{BG}$ ($= h_B$). Figure 1 compares the general format to the restricted Gilbert and Gilbert-Elliott channels.

Starting from general 2-state Markov bit error channels, the same channel 2-state format remains valid for packet errors, whereas the restricted Gilbert and Gilbert-Elliott channels are insufficient for covering the bit as well as the corresponding packet error process.

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Theorem: When the bit stream through a 2-state Markov channel is segmented into a stream of packets of constant length L, then the packet error process is again characterized as a 2-state Markov channel with a state transition after each packet, whose parameters $q^{(L)}$, $p^{(L)}$, $h^{(L)}{}_{GG}$, $h^{(L)}{}_{GB}$, $h^{(L)}{}_{BG}$ and $h^{(L)}{}_{BB}$ can be determined from the set of parameters of the underlying bit error process.

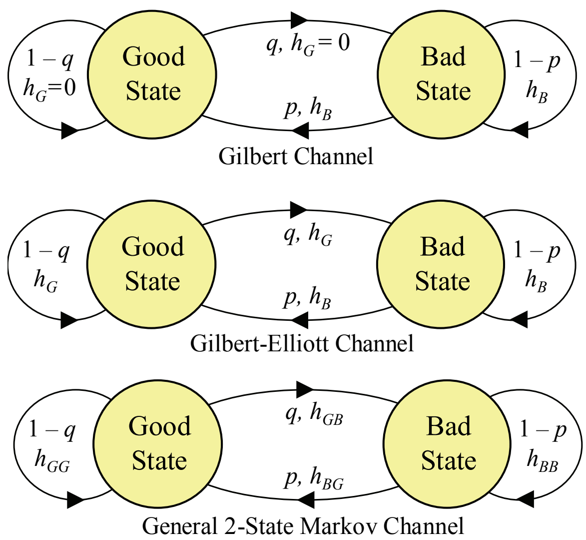

FIGURE 1: GILBERT-ELLIOT & GENERAL 2-STATE MARKOV CHANNELS

For a constructive proof, the parameters of the packet error process are derived by a simple recursive scheme tracing the states of the bit error process over the length L of the packet.

Let g_k^{NoErr}, (b_k^{NoErr}) denote the probability that after the k-th bit of a considered packet ($0 \le k \le L$) the bit error channel is in the "Good" ("Bad") state and no error has occurred on the first k bits of the packet. Then recursive equations determine g_k^{NoErr}, b_k^{0Err} for $k > 0$, which express state transition probabilities from the k^{th} to the $(k+1)^{th}$ bit in the packet under the condition, that no errors occur:

$$\begin{aligned} g_{k+1}^{NoErr} &= (1-q)(1-h_{GG})g_k^{NoErr} + p(1-h_{BG})b_k^{NoErr}; \\ b_{k+1}^{NoErr} &= q(1-h_{GB})g_k^{NoErr} + (1-p)(1-h_{BB})b_k^{NoErr}. \end{aligned} \quad (1)$$

The recursive scheme of equation (1) has the explicit solution:

$$g_k^{NoErr} = \alpha_1\beta_1\gamma_1^k + \alpha_2\beta_2\gamma_2^k; \quad b_k^{NoErr} = \beta_1\gamma_1^k + \beta_2\gamma_2^k; \quad (2)$$

$$\alpha_1 = t_1 + \sqrt{t_1^2 + t_2}; \quad \alpha_2 = t_2 - \sqrt{t_1^2 + t_2};$$

$$\gamma_{1,2} = q(1-h_{GB})\alpha_{1,2} + (1-p)(1-h_{BB});$$

$$t_1 = \frac{(1-q)(1-h_{GG}) - (1-p)(1-h_{BB})}{2q(1-h_{GB})}; \quad t_2 = \frac{p(1-h_{BG})}{q(1-h_{GB})};$$

$$\beta_1 = \frac{g_0^{NoErr} - \alpha_2 b_0^{NoErr}}{\alpha_1 - \alpha_2}; \quad \beta_2 = \frac{\alpha_1 b_0^{NoErr} - g_0^{NoErr}}{\alpha_1 - \alpha_2}.$$

Solution (2) holds for k = 0, 1, 2, …, where especially the parameters β_1 and β_2 are adaptable to arbitrary starting conditions g_0^{NoErr}, b_0^{NoErr}, for example in steady state, i.e. $g_0^{NoErr} = p/(p+q)$ and $b_0^{NoErr} = q/(p+q)$. We obtain the probability of a packet being error-free under steady state conditions

$$p_{(L)}^{NoError} = \left(g_L^{NoErr} + b_L^{NoErr} \mid (g_0^{NoErr}, b_0^{NoErr}) = (\tfrac{p}{p+q}, \tfrac{q}{p+q})\right) = \frac{(1+\alpha_1)(p-q\alpha_2)\gamma_1^L + (1+\alpha_2)(q\alpha_1 - p)\gamma_2^L}{(p+q)(\alpha_1-\alpha_2)}. \quad (3)$$

Gilbert [3] used generating functions for deriving a similar result for the "Recurrence time of ones" in the special case of a Gilbert channel. Equation (2) can be simply proven by

- assuming a geometrical approach $b_k^{NoErr} = \beta\,\gamma^k$; $g_k^{NoErr} = \alpha b_k^{NoErr} = \alpha\,\beta\,\gamma^k$ and
- substitution into both recursive equations (1).

This directly results in a quadratic equation with the solutions α_1, α_2, γ_1, γ_2 as given in equation (2) and leaves freedom to adapt parameters β_1, β_2 to match arbitrary starting conditions for g_0^{NoErr}, b_0^{NoErr}.

In order to fully evaluate the 2-state Markov packet error process and to compute the state specific error rates $h^{(L)}{}_{GG}$ and $h^{(L)}{}_{GB}$ in the "Good" state of the packet error model, we also have to start in the "Good" state. On the other hand, starting in the "Bad" state, we obtain $h^{(L)}{}_{BG}$ and $h^{(L)}{}_{BB}$.

The initialization in the "Good" state (g_0^{NoErr}, b_0^{NoErr}) = (1, 0) yields g_L^{NoErr} as the probability of an error-free packet also ending up in the "Good" state after the last bit. The error probability is then computed as $h^{(L)}{}_{GG} = 1 - g_L^{NoErr}/(1 - q^{(L)})$ where $1-q^{(L)}$ stands for the probability that a packet starts and ends in "Good" state irrespective of errors (g_L^{NoErr}: $Good \xrightarrow{\text{Error-free Packet}} Good$; $1-q^{(L)}$: $Good \xrightarrow{\text{Packet}} Good$).

Other state specific error probabilities are derived analogously

$$\begin{aligned} h^{(L)}{}_{GG} &= 1 - g_L^{NoErr}/(1-q^{(L)}) && \text{start from } (g_0^{NoErr}, b_0^{NoErr}) = (1,0), \\ h^{(L)}{}_{GB} &= 1 - b_L^{NoErr}/q^{(L)} && \text{start from } (g_0^{NoErr}, b_0^{NoErr}) = (1,0), \\ h^{(L)}{}_{BG} &= 1 - g_L^{NoErr}/p^{(L)} && \text{start from } (g_0^{NoErr}, b_0^{NoErr}) = (0,1), \\ h^{(L)}{}_{BB} &= 1 - b_L^{NoErr}/(1-p^{(L)}) && \text{start from } (g_0^{NoErr}, b_0^{NoErr}) = (0,1). \end{aligned} \quad (4)$$

Equations (2-4) are sufficient to compute packet error probabilities $h^{(L)}{}_{GG}$, $h^{(L)}{}_{GB}$, $h^{(L)}{}_{BG}$ and $h^{(L)}{}_{BB}$ depending on the state of the bit error process at the start and at the end of the packet. It remains to compute the state transition probabilities $q^{(L)}$, $p^{(L)}$ of the packet error process, which correspond to the L-step transition matrix $\mathbf{Q}^{(L)}$ of the 2-state Markov bit error process:

$$\mathbf{Q}^{(L)} = \begin{pmatrix} 1-q^{(L)} & q^{(L)} \\ p^{(L)} & 1-p^{(L)} \end{pmatrix} = \mathbf{Q}^L = \begin{pmatrix} 1-q & q \\ p & 1-p \end{pmatrix}^L \quad (5)$$

This again leads to recursive equations and an explicit geometrical solution form

$$\begin{aligned} q^{(k+1)} &= (1-q^{(k)})q + q^{(k)}(1-p); \quad q^{(1)} = q; \\ p^{(k+1)} &= p^{(k)}(1-q) + (1-p^{(k)})p; \quad p^{(1)} = p; \\ q^{(L)} &= q[1-(1-p-q)^L]/(p+q); \\ p^{(L)} &= p[1-(1-p-q)^L]/(p+q). \end{aligned} \quad (6)$$

which completes the construction of the 2-state Markov packet error process.

B. Packet Error Gap and Burst Length Distribution

Equations (1-2) characterize the length of error-free bit sequences. When the bit error channel parameters q, p, h_{GG}, h_{GB}, h_{BG} and h_{BB} are replaced by the packet channel parameters $q^{(L)}$, $p^{(L)}$, $h^{(L)}{}_{GG}$, $h^{(L)}{}_{GB}$, $h^{(L)}{}_{BG}$ and $h^{(L)}{}_{BB}$ then the solution determines the gap length distribution of error-free packet sequences.

It remains to fix the initialization conditions at the start of a gap in steady state [4]. In addition, the solution of equations (1-2) can be modified to determined the burst size of subsequent erroneous packets. Therefore transitions are traced under the condition that errors occur instead of error-free transition runs [4] leading to the same type of recursive equations and solution form by two geometrical distributions. Error bursts are critical for the quality of service, e.g. in voice transmission, where codecs are usually able to conceal single but no bursts of packet errors. While omitting the details of the derivation, we demonstrate the basic results for an example which is continued later on in the consideration of error detection and correction.

C. Evaluation Example

We study packet errors in the example of a 2-state bursty channel with bit error parameters given in the column for L = 1 of Table 1. The channel is characterized by long error-free phases in the "Good" state (mean number of transitions $1/q = 10^5$) which alternate with much shorter "Bad" phases (mean number of transitions $1/p$ = 10). High error probabilities are also associated with state changes from "Good" to "Bad" and vice versa. The analysis of errors by equations (2-6) is carried out for packet error processes with increasing length L = 5, 20, 100, 1000 and an asymptotical result is given for $L \to \infty$.

Table 1 includes packet error probabilities $p^{(L)}{}_{\text{Error}}$ and mean error burst $E[Bst]$ and error gap sizes $E[Gap]$, where $p^{(L)}{}_{\text{Error}} = E[Bst] / (E[Bst]+E[Gap])$.

TABLE 1: EVALUATION OF PACKET ERROR CHANNEL PARAMETERS

Parameters	L=1	L = 5	L = 20	L = 100	L = 1000	$L \to \infty$
$q^{(L)}$	$1 \cdot 10^{-5}$	$4.1 \cdot 10^{-5}$	$8.8 \cdot 10^{-5}$	0.0001	0.0001	0.0001
$p^{(L)} = 10000\, q^{(L)}$	0.1	0.4095	0.8783	0.9999	0.9999	0.9999
$h^{(L)}{}_{GG}$	0	$8.5 \cdot 10^{-6}$	$1.1 \cdot 10^{-4}$	$8.8 \cdot 10^{-4}$	0.00968	1
$h^{(L)}{}_{GB}$	0.8	0.8987	0.9505	0.9566	0.95694	1
$h^{(L)}{}_{BG}$	0.6	0.7974	0.9010	0.9131	0.91388	1
$h^{(L)}{}_{BB}$	0.4	0.9222	0.99996	0.9970	0.99626	1
Packet Error Probability	$5 \cdot 10^{-5}$	$1.3 \cdot 10^{-4}$	$2.8 \cdot 10^{-4}$	0.00107	0.00986	1
Mean Packet Error Gap Size	31728	18898	5065	1018.2	102.26	1
Mean Packet Error Burst Size	1.586	2.503	1.443	1.09	1.019	∞

While packet error probabilities are always monotonously increasing with the length L of a packet, the mean error burst and gap lengths can show different development. In the example the mean error burst size is increasing from L = 1 to L = 5 due to an increase of the error rate $h^{(L)}{}_{BB}$, while those bursts mainly occur in the same "Bad" phase. When L becomes larger (L = 20, …, L = 1000) then it is less likely that a "Bad" phase persists over one or several packets and consequently the burst size is reducing. On the other hand, the mean burst size is increasing unlimited in the asymptotic behaviour, because an increasing number of multiple "Bad" phases are usually encountered in a very long packet, causing all state dependent error probabilities to approach 1.

III. PACKET BASED ERROR DETECTION AND CORRECTION

The previous evaluations assume that packet errors occur as a consequence of bit errors without further means to reduce error rates. However, communication protocols usually protect data transmission against errors by adding redundant information

- for detection and retransmission of erroneous packets or
- for error correction.

The transmission control protocol (TCP) on the transport layer in the Internet applies failure detection and retransmission, presuming that the application is not too delay sensitive. Otherwise, error correcting codes remain as the only alternative when real time applications cannot tolerate delay for retransmission. We study the effect of packet wise error detection or correction in a trade off between the expected reduction of error rates and the additional overhead.

Random error correcting codes are often applied with the capability to correct errors in a packet when the error pattern consists of no more than m single bit errors. Cyclic codes are a usual approach such that the capability to correct an error pattern is valid also for all cyclic shifts of the pattern in a codeword. BCH codes represent a class of cyclic m-error-correcting codes, including Hamming codes for m = 1, where the overhead of redundant control bits per packet is increasing with m [8][9][15].

We evaluate the reduction of packet error rates when m single bit errors are detected or corrected. Based on the 2-state bit error channel, independent block coding of each packet again leads to a 2-state packet error process, with the same parameters $q^{(L)}$, $p^{(L)}$ as computed in equation (6) and with different packet error probabilities $h^{(L)}{}_{GG}$, $h^{(L)}{}_{GB}$, $h^{(L)}{}_{BG}$ and $h^{(L)}{}_{BB}$ specific to the coding scheme.

A. Probability of m Random Errors for 2-State Channels

Let $t_{rs}(m, l)$ denote the probability to encounter m random errors in an l-bit sequence of a 2-state channel starting from state r and ending in s. Using corresponding matrix notation

$$\mathbf{T}(\mu, l) = \begin{pmatrix} t_{GG}(m,l) & t_{GB}(m,l) \\ t_{BG}(m,l) & t_{BB}(m,l) \end{pmatrix}$$

we obtain matrices for single bit transitions with or without errors and a recursive scheme for bit sequences of length l:

$$\mathbf{T}(0, 1) = \begin{pmatrix} (1-q)(1-h_{GG}) & q(1-h_{GB}) \\ p(1-h_{BG}) & (1-p)(1-h_{BB}) \end{pmatrix};$$

$$\mathbf{T}(1, 1) = \begin{pmatrix} (1-q)h_{GG} & qh_{GB} \\ ph_{BG} & (1-p)h_{BB} \end{pmatrix} \quad \text{and} \qquad (7)$$

$$\mathbf{T}(m, l) = \mathbf{T}(m, l-1)\, \mathbf{T}(0, 1) + \mathbf{T}(m-1, l-1)\, \mathbf{T}(1, 1)$$

The latter recursion is reduced to $\mathbf{T}(0, l) = \mathbf{T}(0, l-1)\,\mathbf{T}(0, 1)$ for $m = 0$ corresponding to the explicit formula (2) for error-free sequences. Otherwise, when errors are involved, the computational effort to obtain $\mathbf{T}(m, L)$ for $m = 0, \ldots, \mu$ and for $l = 0, \ldots, L$ is moderate of the order $O(N^3 \mu L)$, where $N = 2$ is the number of states of the considered Markov channel. Thus the recursive scheme can be completely evaluated on a usual personal computer for packet sizes up to $L = 10\,000$ and $\mu = 100$ even for extended Markov channels with more than 2 states.

B. 2-State Packet Error Channel with REC/D Coding

The residual error rates $h_{rs}^{\text{REC/D}}(\mu, L)$ with regard to correction/detection of $\leq \mu$ random errors in a packet of length L while transferring from state r to state s are finally obtained as

$$h_{rs}^{\text{REC/D}}(\mu, L) = 1 - \sum_{m=0}^{\mu} t_{rs}(m, L) / q_{rs}^{(L)} \quad \text{for } r, s \in \{G, B\} \qquad (8)$$

$$\text{where } \mathbf{Q}^{(L)} = \begin{pmatrix} q_{GG}^{(L)} & q_{GB}^{(L)} \\ q_{BG}^{(L)} & q_{BB}^{(L)} \end{pmatrix} = \mathbf{Q}^{L} = \begin{pmatrix} 1-q & q \\ p & 1-p \end{pmatrix}^{L}.$$

The result of equation (8) already covers all state specific residual error rates $h_{rs}^{\text{REC/D}}(\mu, L)$ for $r, s \in \{G, B\}$ as well as the state transition probabilities $p^{(L)}$ and $q^{(L)}$ according to equation (5-6). This complete the set of parameters to describe the 2-state error channel for packets of length L that can tolerate up to μ random errors due to error correcting/detecting codes.

C. Probability for Error Pattern of Length λ in a Packet

Burst error correcting/detecting codes are considered as an alternative, which are able to handle arbitrary error pattern of limited length λ. Burst error coding can be more efficient especially when bit errors are correlated such that several bit errors are often encountered in close neighborhood. A packet is subject to a bit error pattern or burst of length λ , if the positions of the first and last erroneous bit in a packet have a bit distance less than λ, where λ = 0 corresponds to an error free packet, λ = 1 to a single bit error and λ = 2 to a pair of neighboring bit errors. Special cyclic codes have been developed for correcting all error pattern of length ≤ λ [8][9][11][15]. Cyclic codes for detection of bursts of length ≤λ can be simply based on λ independent parity check bits, see Section IV.

For an efficient analysis we compute the probabilities $b_{r,s}(k, \lambda)$ of a set of error pattern over a packet of length L which

- has a bit error at the k^{th} position of the packet
- includes an error burst of length l in the range $1 \leq l \leq \lambda$
- switches from state r at the start to state s at the end of the packet $r,s \in \{G, B\}$ = {"Good", "Bad"}.

When we denote an erroneous bit by "1" and a correct bit by "0" in a bit error pattern then this set of bit error pattern has the presentation

$$0^{k-1}\,1\,\{0 \text{ or } 1\}^{\lambda-1}\,0^{L-k-\lambda+1}.$$

An erroneous bit at position k is always included in order to have unique and disjoint subsets of error burst. Then we obtain in matrix notation

$$\mathbf{B}(k,\lambda) = \begin{pmatrix} b_{GG}(k,\lambda) & b_{GB}(k,\lambda) \\ b_{BG}(k,\lambda) & b_{BB}(k,\lambda) \end{pmatrix} = \mathbf{T}(0, k-1)\,\mathbf{T}(1,1)\,\mathbf{Q}^{(\lambda-1)}\mathbf{T}(0, L-k-\lambda+1). \qquad (9)$$

The computational effort of an efficient implementation is of the order $O(N^3)$ where all matrices are explicitly defined in equations (2), (5-6) and have also been used in section III *A*.

D. 2-State Packet Error Channel with BEC/D Coding

We finally compute the residual error rates $h_{rs}^{\text{BEC/D}}(\lambda)$ with regard to correction/detection of error bursts of length ≤ λ starting at arbitrary positions k in a packet of length L from the previous result

$$h_{rs}^{\text{BEC/D}}(\lambda) = 1 - \left(t_{rs}(0, L) + \sum_{k=1}^{L-\lambda+1} b_{rs}(k,\lambda)\right) / q_{rs}^{(L)} \qquad (10)$$

where $t_{rs}(0, L)$ denotes the case of an error-free packet starting from state r and ending in s. Computation of $h_{rs}^{\text{BEC/D}}(\lambda)$ for $r, s \in \{G, B\}$ as well as for $\lambda = 0, \ldots, \lambda_{\max}$ has a computational complexity of the order $O(N^2 \lambda_{\max} L)$. The entire analysis for residual error rates of length ≤ λ is no more complex as the analysis of for ≤ λ random errors as derived in Section III *A-B*.

E. Performance of Random versus Burst Error Correction

For comparing residual packet error rates with coding we applied the previous analysis methods to packets of length $L = 1000$ with bit channel parameters as in Table 1, for $L = 1$. In a final step we obtain the residual error rate of a code from the state specific rates assuming steady state conditions:

$$h^{\text{REC/D}}(N) = \frac{p(h_{GG}^{\text{REC/D}}(N,L) + h_{GB}^{\text{REC/D}}(N,L))}{p+q} + \frac{q(h_{BG}^{\text{REC/D}}(N,L) + h_{BB}^{\text{REC/D}}(N,L))}{p+q} \qquad (11)$$

$$h^{\text{BEC/D}}(N) = \frac{p(h_{GG}^{\text{BEC/D}}(N) + h_{GB}^{\text{BEC/D}}(N)) + q(h_{BG}^{\text{BEC/D}}(N) + h_{BB}^{\text{BEC/D}}(N))}{p+q}$$

Figure 2 shows the results regarding the capability to correct or to detect and correctly repeat

- all packets subject to ≤ N random bit errors and
- all packets subject to any bit error pattern of length ≤ N.

Figure 2 indicates a slow decrease of the error rate with N. The capability to correct a few random errors yields a much faster decrease in the residual error rates down below 10^{-10} for uncorrelated binary symmetrical channels.

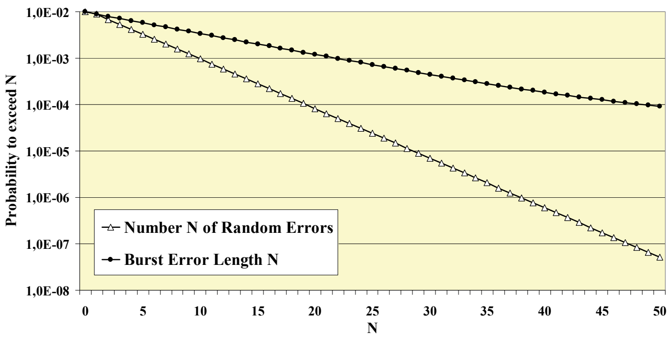

FIGURE 2: RESIDUAL ERROR RATE DUE TO > N ERRORS OR BURSTS OF LENGTH > N IN THE 2-STATE MARKOV CHANNEL EXAMPLE OF TABLE 1

However, the considered example of a bursty 2-state error channel is often more realistic but makes it much more difficult to reduce error rates. For a reduction down to 10^{-4}, a code has to correct 20 random errors or all bit error patterns up to length 49.

F. 2-State Packet Error Processes with Regard to Coding

The 2-state Markov bit error channel again implies a 2-state packet error process when coding is done for each packet independent of other packets. In sections III *B* and III *D* we already derived the state specific error rates and have completely determined the parameter set of the 2-state Markov packet error processes with regard to codes applicable to random and burst errors, respectively. Table 2 shows parameter sets $h_{rs}^{\mathrm{REC/D}}(20, 1000)$ and $h_{rs}^{\mathrm{BEC/D}}(49)$ for two codes capable

- to correct (or to detect) up to 20 random errors and
- to correct (or to detect) errors of burst lengths up to 49.

Both codes achieve comparable residual error rates slightly below 10^{-4}, where all error patterns of ≤ 20 random errors (burst length ≤ 49) are assumed to be handled correctly and all other error pattern are subsumed under the residual error rate. Error correction and error detection codes for the same residual error rate differ in the overhead of redundant control bits as discussed in detail in Section IV.

TABLE 2: 2-STATE MARKOV PACKET ERROR PROCESSES DETERMINED BY STATE SPECIFIC RESIDUAL ERROR RATES OF CODES

Parameters of 2-state Markov packet error process with coding	20 Random Error Correction/Detection	Burst Length 49 Error Correction/Detection
$q^{(1000)}$ (bit error channel of table 1)	0.0001	0.0001
$p^{(1000)} = 10000\ q^{(1000)}$	0.9999	0.9999
$h_{GG}^{\mathrm{REC/D}}(20, 1000)$ | $h_{GG}^{\mathrm{BEC/D}}(49)$	0.00008043	0.00009509
$h_{GB}^{\mathrm{REC/D}}(20, 1000)$ | $h_{GB}^{\mathrm{BEC/D}}(49)$	0.00753	0.0136
$h_{BG}^{\mathrm{REC/D}}(20, 1000)$ | $h_{BG}^{\mathrm{BEC/D}}(49)$	0.00719	0.0130
$h_{BB}^{\mathrm{REC/D}}(20, 1000)$ | $h_{BB}^{\mathrm{BEC/D}}(49)$	0.04449	0.8751
Residual Error Rate	0.00008188	0.00009774

The results in Table 2 show that state specific error rates are low when the "Good" state is held but essentially increase when a "Bad" state is involved at the beginning or end of a packet. A 20-random-error-correcting/detecting code is less efficient in cases of a long "Bad" phase generating more than 20 errors. A burst length 49 correcting/detecting code reduces the residual error rate for long bursts, but cannot deal with several "Bad" phases in the same packet, which are more than 50 bits apart from each other. The code for 20 random errors even can correct most of the packets including two "Bad" phases since the mean length of a "Bad" phase is 10 and the error rate is 0.4 during "Bad" state, such that the mean number of errors in a "Bad" phase is 4.

We have compared codes with total residual error rate close to 10^{-4}, such that there is not much difference in the most relevant specific error rate of the "Good" state. On the other hand, when packets start and end in a "Bad" phase then a code for random errors still can handle many cases of short error bursts at both ends while burst error codes fail except for a few cases where no bit error occurs at one end of the packet. Thus the differences are large when the bad state holds over a packet.

IV. OVERHEAD OF RANDOM & BURST ERROR CORRECTION

Finally, the overhead in terms of the fraction of control bits per packet is considered for the following cases:

A. Correction of up to m random errors

In order to correct m errors in a packet of length L, the control information has to distinguish all possible combinations of error positions in the packet, such that different pattern are generated as the control information in each case. From this fact we obtain a lower bound on the required length $c_{REC}(m)$ of the control information, i.e. the number of redundant bits that have to be added, where $c_{REC}(m)$ is included in the packet length L:

$$c_{REC}(m) \ge \log_2 \sum_{j=0}^{m} \binom{L}{j}$$

From a different perspective, the distance between codewords in terms of the number of differing bits must be at least $2m+1$ in order to enable the correction of m errors. A distance of $2m+1$ means that there are disjoint error correction spaces of radius m around each codeword, such that a unique reassignment of those error pattern is possible which stay in the correction space of radius m around the original codeword [8][9]. This leads to the same bound on the control information.

B. Detection of up to m random errors

In principle, the same type of codes can be used for error correction as well as for error detection, where an error is detected if the received packet or data block is not a codeword. Then a code with (minimum) distance $m+1$ obviously allows detecting all error pattern of up to m bits. If m is even and with $m + 1 = 2\ k + 1$, such a code could also correct k errors. Therefore we can transfer the previous bound on the length $c_{REC}(k)$ of control information for correction to a bound on the control information length $c_{RED}(m)$ for detection, where again $c_{REC}(m)$ is included in the packet length L:

$$c_{RED}(m) \ge c_{REC}(m/2) \ge \log_2 \sum_{j=0}^{m/2} \binom{L}{j} \quad \text{for even } m.$$

If m is odd, then it is sufficient to add a parity check bit in order to increment to an even distance $m+1$, such that

$c_{RED}(m) = c_{RED}(m-1) + 1$ for odd m.

C. Correction of Burst Errors up to Length λ

In order to determine the number of different error pattern of length λ in a packet of length L, we have to consider the bit positions $p = 1, \ldots, L - \lambda + 1$ as possible starting points of the burst. The bit pattern of a burst of length λ starts and ends with an error and may have any arbitrary bit sequence in between. Thus there are $(L - \lambda + 1)\cdot 2^{\lambda-2}$ different burst pattern of length λ in a packet and we obtain the lower bound $c_{BEC}(\lambda)$:

$$c_{BEC}(\lambda) \geq \log_2\left(L + \sum_{j=2}^{\lambda}(L - j + 1)\cdot 2^{j-2}\right)$$

Once more the $c_{BEC}(\lambda)$ control bits are assumed to be included in the packet length L.

D. Detection of Burst Errors up to Length λ

A code with λ control bits can be set up to detect all error pattern of length ≤ λ. Therefore λ parity check bits are introduced such that

- the first parity check includes bit positions 1, λ+1, 2λ+1, 3λ+1, …
- the second parity check includes bit positions 2, λ+2, 2λ+2, 3λ+2, …
- …
- the λ^{th} parity check includes bit positions λ, 2λ, 3λ, …

Then an error pattern of length ≤ λ will affect no more than one bit in each of those parity checks and thus can be discovered. Those parity checks can be combined with a cyclic code with $\log_2 L$ additional control bits for the purpose of error correction in the previous case. For detection we conclude that $c_{BED}(\lambda) = \lambda$ control bits are sufficient for burst length λ, which are again covered in the packet of length L.

E. Comparative Overhead Evaluation

We compare the previously derived estimates on the overhead of coding schemes for

- random error correction
  $c_{REC}(m) \geq \log_2 \sum_{j=0}^{m}\binom{L}{j}$,
- random error detection
  $c_{RED}(m) \geq \log_2 \sum_{j=0}^{m/2}\binom{L}{j}$ for even m,
  $c_{RED}(m) = c_{RED}(m-1)+1$ for odd m,
- burst error correction
  $c_{BEC}(\lambda) \geq \log_2\left(L + \sum_{j=2}^{\lambda}(L - j + 1)\cdot 2^{j-2}\right)$,
- and burst error detection $c_{BED}(\lambda) = \lambda$

for packet length $L = 1000$. The results are shown in Figure 4.

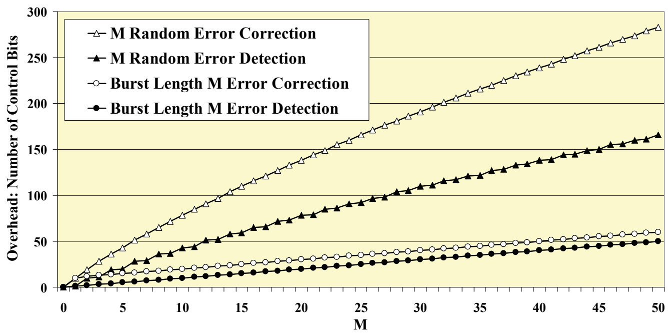

FIGURE 4: REQUIRED NUMBER OF CONTROL BITS TO SECURE PACKETS OF LENGTH $L = 1000$

In continuation of the example in Table 1 and Figure 2, a 20-random-error-correcting code can reduce the error rate down to 10^{-4} by introducing an overhead of at least 138 control bits, which corresponds to 13.8% of the packet length. A 20-random-error-detecting code can achieve the same error rate with 7.8% overhead, provided that a foreseen retransmission is successful. In fact, both overhead fractions are lower bounds which may be exceeded by a concrete coding scheme, but there are codes that can closely approach those bounds [8][9][15].

Regarding correction and detection of error bursts of length λ, the overhead corresponds to codes which follow a simple construction principle for that purpose [8][15]. The overhead is λ/L in case of error detection and $(\lambda + \log_2 L) / L$ for error correction. In the example, a burst length of $\lambda = 49$ is required to reduce the error rate down to 10^{-4}. Thus burst-length-49 correcting or detecting codes achieve comparable performance as the 19-random-error correcting or detecting codes, while their overhead is lower at 5.9% for correction and 4.9% for detection of errors. Therefore burst error correcting codes are more efficient when a residual error rate of 10^{-4} is demanded for the bursty 2-state Markov error channel given in table 1. In general, it depends on the parameters of the channel whether random or bursty error pattern are mixed or one type is more dominant.

F. Remark on Applications for Variable Length Packets

Assuming variable length packets, such that each packet has length k with i.i.d. probability $p(L = k)$, the packet error process still is covered by a 2-state Markov channel. The parameters $q^{(VL)}$, $p^{(VL)}$, $h^{(VL)}_{GG}$, $h^{(VL)}_{GB}$, $h^{(VL)}_{BG}$ and $h^{(VL)}_{BB}$ are then determined as a weighted sum over the results for fixed length packets:

$$q^{(VL)} = \sum_k p(L=k)q^{(k)}; \quad p^{(VL)} = p(L=k)p^{(k)};$$

$$h^{(VL)}_{st} = \sum_k p(L=k)h^{(k)}_{st} \quad \text{for } s,t \in \{G,B\}.$$

Many communication protocols generate variable length packets including the Internet protocol (IP). Thus in principle the results can be extended from fixed packet length to i.i.d. packet length distributions.

V. Conclusion and Outlook

An exact performance analysis of residual errors of packet transmission with error correction and detection is feasible for 2-state Markov channels based on recursive and partly on explicit solutions. The packet error process is again characterized as a 2-state Markov process involving six parameters for the state transitions and the state specific error rates, which are determined from the bit error process of the same type, including Gilbert and Gilbert-Elliot channels as special cases.

We have evaluated examples to compare random and burst error correction and detection schemes with regard to the corresponding overhead using generally applicable and tractable calculation methods. The 2-state Markov analysis is widely applicable in order optimize the packet length in transmission protocols with regard to demanded residual error rates including coding for error correction and retransmission schemes.

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